Optimal consumption and investment strategies with stochastic interest rates

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Abstract

We characterize the solution to the consumption and investment problem of a power utility investor in a continuous-time dynamically complete market with stochastic changes in the opportunity set. Under stochastic interest rates the investor optimally hedges against changes in the term structure of interest rates by investing in a coupon bond, or portfolio of bonds, with a payment schedule that equals the forward-expected (i.e. certainty equivalent) consumption pattern. Numerical experiments with two different specifications of the term structure dynamics (the Vasicek model and a three-factor non-Markovian Heath–Jarrow–Morton model) suggest that the hedge portfolio is more sensitive to the form of the term structure than to the dynamics of interest rates.
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1. Introduction

Since the pathbreaking papers of Merton (1969, 1971, 1973) it has been recognized that long-term investors want to hedge stochastic changes in investment opportunities, such as changes in interest rates, excess returns, volatilities, and inflation rates. The main contribution of this paper is to enhance the understanding of how investors with

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constant relative risk aversion (CRRA) preferences for consumption (and, possibly, terminal wealth) should optimally hedge interest rate risk. We demonstrate that the optimal hedge against changes in interest rates is obtained by investing in a coupon bond, or portfolio of bonds, with a payment schedule that precisely equals the certainty equivalents of the future optimal consumption rates. Furthermore, we study the importance for interest rate hedging of both the current form and the dynamics of the term structure. In a numerical example we compare the solutions for a standard one-factor Vasicek and a three-factor model where the term structure can exhibit three kinds of changes: a parallel shift, a slope change, and a curvature change. Our findings suggest that the form of the initial term structure is of crucial importance for the optimal future consumption plan and, hence, important for the relevant interest rate hedge, while the specific dynamics of the term structure is of minor importance.

As shown by Heath et al. (1992), any dynamic interest rate model is fully specified by the current term structure and the forward rate volatilities. Therefore, the Heath–Jarrow–Morton (HJM) modeling framework is natural for the purpose of comparing the separate effects of the current term structure and the dynamics of the term structure on the optimal interest rate hedging strategy. The HJM class nests all Markovian interest rate models, such as the Vasicek model. However, models outside this Markovian class also frequently arise within the HJM modeling framework. This is, for example, the case for the three-factor model considered in our numerical example.

Given that we want to compute optimal investment strategies in possibly non-Markovian models, we first derive a general, exact characterization of both optimal consumption and portfolio choice in a framework that also allows for non-Markovian dynamics of asset prices and the term structure of interest rates, but requires dynamically complete markets. This characterization generalizes recent results in specialized Markovian settings (Liu, 1999; Wachter, 2002a). For the special case where interest rates have Gaussian, but still potentially non-Markovian, HJM dynamics, we obtain the explicit solution for the optimal consumption and investment strategies that we use for studying the impact of the current form and the dynamics of the term structure on hedging demand. To our knowledge, this paper provides the first explicit solution to an intertemporal consumption and investment problem where the dynamics of the opportunity set is non-Markovian and the investor has non-logarithmic utility.

There has recently been a number of studies of optimal investment strategies with specific assumptions on the dynamics of interest rates. Brennan and Xia (2000) and Sørensen (1999) consider the investment problem of a CRRA utility investor with utility from terminal wealth only. They assume complete markets and show that in the case where the term structure of interest rates is described by a Vasicek-type model and market prices on risk (and expected excess returns) are constant, the optimal hedge portfolio is the zero-coupon bond that expires at the investment horizon. This particular result is also obtained as a special case within the framework of this paper. Liu (1999) provides similar insight using the one-factor square-root model of Cox et al. (1985).

A few papers have addressed the portfolio problem under stochastic interest rates for investors with utility over consumption. In a general complete-market setting,

Wachter (2002b) shows that an infinitely risk-averse agent will only invest in a coupon bond. This result is also a special case of our findings, but we solve for the optimal portfolio for CRRA investors with any level of risk aversion. Both Campbell and Viceira (2001) and Brennan and Xia (2002) study consumption and portfolio choice problems in settings with uncertain inflation, where real interest rates follow a one-factor Vasicek model. While we ignore inflation risk, we allow for more general dynamics of interest rates. [1]

The general modeling of the investment opportunity set in this paper nests the Markovian models studied in the above-mentioned papers. Furthermore, we explicitly link the optimal hedge portfolio to the optimal consumption pattern of the investor. In addition, we study how sensitive the optimal hedge against interest rate risk is to the current form of the term structure and to the dynamics of the term structure of interest rates.

The rest of the paper is organized as follows. In Section 2 we set up the general continuous-time consumption and investment problem in a dynamically complete market and provide a general characterization of the optimal consumption and investment policy for a CRRA investor in a possibly non-Markovian setting. In Section 3 we derive explicit results showing how to hedge against changes in the term structure of interest rates using coupon bonds in a specialized HJM multi-factor Gaussian term structure setting. In Section 4 we consider two specific numerical examples based on the Vasicek model and an HJM three-factor model. We compare the hedge bonds in the two examples for different levels of risk aversion and different forms of the initial term structure of interest rates. Section 5 concludes. [2]

2. Portfolio choice with general dynamics in investment opportunities

We consider a frictionless economy where the dynamics is generated by a d-dimensional Wiener process, $w = (w_1, \ldots, w_d)$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. $F = \{\mathscr{F}_t : t \geqslant 0\}$ denotes the standard filtration of w and, formally, $(\Omega, \mathscr{F}, F, \mathbb{P})$ is the basic model for uncertainty and information arrival in the following.

2.1. Preferences

We will consider the investment problem of an expected utility maximizing investor with a time-separable constant relative risk aversion utility function given by

$$K \cdot E_0 \left[\int_0^T U_1(C_t, t) \, \mathrm{d}t \right] + (1 - K) \cdot E_0[U_2(W_T)], \tag{1}$$

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[1] Inflation can be introduced along the same lines in the set-up of this paper in which case the relevant bond for hedging purposes would be an indexed bond with payments that in real terms match the forward-expected consumption pattern.

[2] Proofs and detailed derivations are contained in an appendix which is available from the authors by request.

where

$$U_1(C,t) = e^{-\beta t}\frac{C^{1-\gamma} - 1}{1 - \gamma},$$

$$U_2(W) = e^{-\beta T}\frac{W^{1-\gamma} - 1}{1 - \gamma},$$

and where β is a constant subjective time discount rate and γ is a constant relative risk aversion parameter. The preference parameter K controls the relative weight of intermediate consumption, C_t, and terminal wealth, W_T, in the agent's utility function. The special case where $\gamma = 1$ is the logarithmic utility case: $U_1(C,t) = e^{-\beta t}\log C$ and $U_2(W) = e^{-\beta T}\log W$.

2.2. Investment assets

The agent can invest in a set of financial securities. One of these financial assets is assumed to be an ''instantaneously'' risk-free bank account which has a return equal to the short-term interest rate r_t. In addition, the agent can invest in d risky assets with prices described by the vector $V_t = (V_{1t}, \ldots, V_{dt})'$. The price dynamics of the risky assets (cum dividend) is given by

$$dV_t = \text{diag}(V_t)[(r_t 1_d + \sigma_t\lambda_t)\,dt + \sigma_t\,dw_t], \tag{2}$$

where λ_t is an $\mathbb{R}^d$-valued stochastic process of market prices of risk, σ_t is an $\mathbb{R}^{d\times d}$-valued stochastic process of volatilities, 1_d is a d-dimensional vector of ones, and $\text{diag}(V_t)$ is a $(d \times d)$-dimensional matrix with V_t in the diagonal and zeros off the diagonal. It is assumed that σ has full rank d implying that markets are dynamically complete (cf. Duffie and Huang, 1985). As a consequence of markets being dynamically complete, the pricing kernel (or state-price deflator) is uniquely determined and given by (see, e.g., Duffie, 1996, Chapter 6)

$$\zeta_t = \exp\left\{ -\int_0^t r_s\,ds - \int_0^t \lambda_s'\,dw_s - \frac{1}{2}\int_0^t \|\lambda_s\|^2\,ds \right\}, \quad t \geqslant 0, \tag{3}$$

or, equivalently, in differential form,

$$d\zeta_t = \zeta_t[-r_t\,dt - \lambda_t\,dw_t], \quad \zeta_0 = 1. \tag{4}$$

The present value of any stochastic payoff, X, paid at some future time point s can be determined by evaluating the pricing-kernel-weighted payoff. In particular, we have

$$\text{PV}_t[X] = E_t\left[\left(\frac{\zeta_s}{\zeta_t}\right)X \right] = P_t(s)\widehat{E}_t^s[X], \tag{5}$$

where $P_t(s)$ is the time t price on a zero-coupon bond that expires at time s. The last equality defines the so-called certainty-equivalent or forward-expected payoff, $\widehat{E}_t^s[X]$; see, e.g., Jamshidian (1987, 1989) and Geman (1989) who introduce the notion of the forward risk-neutral martingale measure, as being distinct from the usual risk-neutral martingale measure in the context of interest rate models.

2.3. The problem and the general solution

Let π_t be an $\mathbb{R}^d$-valued process describing the fractions of wealth that the agent allocates into the d different risky assets. The wealth of the agent then evolves according to

$$\mathrm{d}W_t = [(r_t + \pi_t'\sigma_t\lambda_t)W_t - C_t]\,\mathrm{d}t + W_t\pi_t'\sigma_t\,\mathrm{d}w_t. \tag{6}$$

The agent's problem is to choose a dynamic consumption strategy, C_t, and portfolio policy, π_t, in order to maximize the expected utility in (1). This problem has traditionally been addressed and solved by using a dynamic programming approach, cf. Merton (1969, 1971, 1973). The main idea of the martingale solution approach suggested and formalized by Cox and Huang (1989, 1991) and Karatzas et al. (1987) is to alternatively consider the static problem

$$\sup_{\{C_t,W_T\}} K \cdot E_0\left[\int_0^T U_1(C_t,t)\,\mathrm{d}t\right] + (1-K) \cdot E_0[U_2(W_T)] \tag{7}$$

subject to

$$E_0\left[\int_0^T \left(\frac{\zeta_t}{\zeta_0}\right)C_t\,\mathrm{d}t + \left(\frac{\zeta_T}{\zeta_0}\right)W_T\right] \leqslant W_0. \tag{8}$$

In principle, the problem is to maximize expected utility subject to the budget constraint (8), which states that the present value of the consumption stream and terminal wealth cannot exceed the agent's current wealth. As shown by Cox and Huang (1989, 1991) and Karatzas et al. (1987), the solution to this problem also provides the solution to the dynamic problem of choosing the optimal consumption strategy and portfolio policy. The value function, or indirect utility, J_t, from the optimization problem is the maximum expected remaining life-time utility which can be achieved by the optimal consumption and terminal wealth plan following any time point t, $0 \leqslant t \leqslant T$.

The problem in (7) and (8) is a standard Lagrangian optimization problem which can be solved using the Saddle Point Theorem (see, e.g., Duffie, 1996, pp. 205–208) to determine the optimal consumption process, C_t, and terminal wealth, W_T. Thus, under the specific CRRA utility assumption in (1), the optimal consumption plan given information available at time t takes the form [3]

$$C_s = \frac{W_t}{Q_t}K^{\frac{1}{\gamma}}\mathrm{e}^{-\frac{\beta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{-\frac{1}{\gamma}}, \quad 0 \leqslant t \leqslant s \leqslant T, \tag{9}$$

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[3] The details in this derivation are contained in the appendix which is available from the authors by request.

where the (investor-specific) stochastic process Q_t is defined by

$$Q_t = K^{\frac{1}{\gamma}} \int_t^T e^{-\frac{\beta}{\gamma}(s-t)} E_t \left[\left(\frac{\zeta_s}{\zeta_t} \right)^{\frac{\gamma-1}{\gamma}} \right] ds + (1-K)^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} E_t \left[\left(\frac{\zeta_T}{\zeta_t} \right)^{\frac{\gamma-1}{\gamma}} \right]. \tag{10}$$

Note that the current consumption rate at time t is given by $C_t = (W_t/Q_t)K^{\frac{1}{\gamma}}$ and, hence, that Q_t describes the wealth-to-consumption ratio. As formalized in Proposition 1 below, Q_t is also crucial for determining how to hedge against changes in the opportunity set.

While the consumption policy is usually given explicitly by solving (7) and (8), as in (9), the optimal portfolio policy is only given implicitly as the policy which replicates the optimal terminal wealth from the above problem and in accordance with (6). The existence and uniqueness of such a portfolio policy follow from the Martingale Representation Theorem (see, e.g., Duffie, 1996). [4]

For log-investors ($\gamma = 1$) it is well known that the optimal portfolio is the growth-optimal portfolio, but in order to derive an explicit expression for the optimal portfolio for other investors it is generally recognized that the price dynamics must be specialized. Cox and Huang (1989) show that when the state-price deflator and the risky asset prices constitute a Markovian system, the optimal investment strategy can be represented in terms of the solution of a linear second-order partial differential equation. On the other hand, the following proposition provides a closed-form expression for the optimal investment strategy for a power utility investor in a general possibly non-Markovian complete market setting for a CRRA investor.

Since Q_t, as defined in (10), is a positive stochastic process adapted to the filtration generated by w_t, it follows from the Martingale Representation Theorem, that the dynamics of Q_t can be described by

$$dQ_t = Q_t[\mu_{Qt} \, dt + \sigma_{Qt} \, dw_t] \tag{11}$$

for some drift process μ_{Qt} and some volatility process σ_{Qt}. The precise forms of μ_{Qt} and σ_{Qt} depend on the specific assumptions on the pricing kernel and, subsequently, we will consider such specific examples and apply the following general proposition.

Proposition 1. *The value function of the general problem in* (7) *and* (8) *has the form*

$$J_t = \frac{Q_t^{\gamma} W_t^{1-\gamma} - A(t)}{1-\gamma}, \tag{12}$$

where

$$A(t) = \frac{K}{\beta}(1 - e^{-\beta(T-t)}) + (1-K)e^{-\beta(T-t)}$$

and Q_t is defined in Eq. (10).

The optimal consumption choice and the optimal portfolio policy at time t are given by

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[4] The optimal investment strategy can be represented rather abstractly in complete markets in terms of stochastic integrals of Malliavin derivatives by the Clark–Ocone formula (cf. Ocone and Karatzas, 1991).

$$C_t = K^{\frac{1}{\gamma}}\frac{W_t}{Q_t} \tag{13}$$

and

$$\pi_t = \left(\frac{1}{\gamma}\right)(\sigma_t')^{-1}\lambda_t + (\sigma_t')^{-1}\sigma_{Qt}. \tag{14}$$

Proof. The proof is available from the authors upon request. $\quad\square$

Proposition 1 states the optimal consumption and investment strategies in our setting that allows for general, possibly non-Markovian, shifts in the investment opportunity set. However, a special case is the Markovian setting where the shifts in investment opportunities are governed by a Markov diffusion process x with dynamics

$$dx_t = \mu_x(x_t, t)\,dt + \sigma_x(x_t, t)\,dw_t.$$

In this case, the basic optimization problem considered in this paper could alternatively be solved using a traditional dynamic programming approach, and it is well known that some (unknown) function $Q(x_t, t)$ exists such that J_t is given as in (12) with Q_t replaced by $Q(x_t, t)$ (see, e.g., Ingersoll, 1987). In this case the characterization of optimal consumption in (13) follows from the so-called envelope condition and, furthermore, it follows directly by Ito's lemma that σ_{Qt} in (14) can be characterized on the form $\sigma_x(x_t, t)\frac{\partial Q}{\partial x}/Q(x_t, t)$. Proposition 1 provides an explicit characterization of the function Q and, in particular, extends the result so that it also applies for non-Markovian market settings where a dynamic programming approach does not directly apply.

As in Merton (1971), the portfolio policy can be decomposed into a speculative portfolio (the first term in (14)) and a hedge portfolio that describes how the investor should optimally hedge against changes in the investment opportunity set (the last term in (14)). The investor must thus form a hedge portfolio that basically mimics the dynamics of Q_t and, hence, Q_t reflects everything of importance for how to hedge against changes in the investment opportunity set. For a given investor it can be inferred from (10) that only processes included in the description of (moments of) the pricing kernel stated in (3) are relevant for intertemporal hedging purposes. In general, the investor should alone consider hedging against changes in interest rates and changes in prices on risk in the economy while changes in, say, volatilities on marketed securities should be of no concern in our complete market setting.

It is instructive to consider two special cases: the log-utility case ($\gamma = 1$) and the case of an infinitely risk-averse investor ($\gamma = \infty$). [5] The log-utility investor does not hedge against changes in the opportunity set at all (the last term in (14) vanishes as $\gamma \to 1$) and the optimal consumption rate is $C_t = KW_t/A(t)$, i.e. a time-varying, but

[5] Formally, the results for an infinitely risk averse investor are defined as the limiting results of Proposition 1 as $\gamma \to \infty$.

deterministic, fraction of wealth. The infinitely risk-averse investor has no speculative demand for securities at all (the first term in (14) vanishes as $\gamma \to \infty$). If this investor has utility from both consumption and terminal wealth, the Q-process reduces to

$$Q_t = \int_t^T P_t(s)\,\mathrm{d}s + P_t(T). \tag{15}$$

Hence, the hedge portfolio is an annuity bond. (In the special case where the investor has utility from terminal wealth only, i.e. $K = 0$, the hedge portfolio is a zero-coupon bond that expires at the investment horizon.) According to (9), the optimal consumption strategy is in this case constant, $C_t = W_t/Q_t = W_0/Q_0$, and the optimal consumption strategy is thus basically implemented by using the certain payments on the annuity bond for consumption. [6]

3. Hedging changes in interest rates

In the rest of the paper we focus on how to hedge changes in interest rates. In this section we will provide an explicit solution to the consumption and investment choice problem when interest rates evolve according to a HJM model. [7] This is an application of Proposition 1. Furthermore, we demonstrate a close link between the hedging demand and the optimal consumption stream.

For convenience and clarity we separate the investment assets into stocks and bonds in the following. Formally, we split the d-dimensional Wiener process generating the financial asset returns as $w = (w_B, w_S)$, where w_B is of dimension k and w_S is of dimension $l = d - k$. We assume that the dynamics of the term structure of interest rates, and, hence, the dynamics of prices on bonds and other term structure derivatives traded at the bond market, are affected only by w_B. The dynamics of the stock prices may depend on both w_B and w_S which allow for correlation between stocks and term structure derivatives. Specifically, the investor can invest in the "instantaneously" risk-free bank account, k term-structure derivatives, and l stocks. The asset price dynamics is given by

$$\mathrm{d}B_t = \mathrm{diag}(B_t)[(r_t 1_k + \sigma_{Bt}\lambda_{Bt})\,\mathrm{d}t + \sigma_{Bt}\,\mathrm{d}w_{Bt}] \tag{16}$$

and

$$\mathrm{d}S_t = \mathrm{diag}(S_t)[(r_t 1_l + \varphi_{St})\,\mathrm{d}t + \sigma_{S1t}\,\mathrm{d}w_{Bt} + \sigma_{S2t}\,\mathrm{d}w_{St}], \tag{17}$$

where σ_B, σ_{S1}, and σ_{S2} are matrix valued processes of dimension $k \times k$, $l \times k$, and $l \times l$, respectively. Again, σ_B and σ_{S2} are assumed non-singular so that markets are complete. Changes in the returns of the term structure derivatives and the stocks are

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[6] An annuity bond is a coupon bond where the certain cash flows (coupon + principal repayment) from the bond are the same throughout the finite life of the bond.

[7] The HJM approach is, to our knowledge, the most general interest rate modeling framework, and any term structure model that does not allow for arbitrage can be represented in a HJM setting.

correlated with $k \times l$ covariance matrix $\sigma_{Bt}\sigma'_{S1t}$. The market price of risk process λ (which is not dependent on the particular set of assets chosen) has the form

$$\lambda_t = \begin{pmatrix} \lambda_{Bt} \\ \lambda_{St} \end{pmatrix},$$

where

$$\lambda_{St} = \sigma_{S2t}^{-1}\varphi_{St} - \sigma_{S2t}^{-1}\sigma_{S1t}\lambda_{Bt}.$$

Note that we have introduced the $\mathbb{R}^l$-valued stochastic process φ_{St} ($= \sigma_{S1t}\lambda_{Bt} + \sigma_{S2t}\lambda_{St}$), which can be interpreted as the expected excess return on the stocks.

More specifically, we assume that the dynamics of the term structure of interest rates can be described by a k-factor model of the HJM class introduced by Heath et al. (1992). For any maturity date τ the dynamics of the τ-maturity instantaneous forward rate is

$$f_t(\tau) = f_0(\tau) + \int_0^t \alpha(s,\tau)\,\mathrm{d}s + \int_0^t \sigma_f(s,\tau)'\,\mathrm{d}w_{Bs}, \tag{18}$$

where $\sigma_f(\cdot,\tau)$ is an $\mathbb{R}^k$-valued deterministic function and $f_0(\tau)$ is the τ-maturity forward rate observed initially at time 0. The short-term interest rate is $r_t = f_t(t)$. As a no-arbitrage drift restriction we have that

$$\alpha(t,\tau) = \sigma_f(t,\tau)'\left(\lambda_B(t) + \int_t^\tau \sigma_f(t,u)\,\mathrm{d}u\right),$$

so that one only has to specify the initial term structure of forward rates and the volatility structure $\sigma_f(t,\tau)$.

Among the many term-structure derivatives, we focus on default-free bonds. The dynamics of the price $P_t(\tau) = \exp(-\int_t^\tau f_t(s)\,\mathrm{d}s)$ of the zero-coupon bond maturing at time τ is given by

$$\mathrm{d}P_t(\tau) = P_t(\tau)[(r_t + \sigma_P(t,\tau)'\lambda_B(t))\,\mathrm{d}t + \sigma_P(t,\tau)'\,\mathrm{d}w_{Bt}], \tag{19}$$

where $\sigma_P(t,\tau) = -\int_t^\tau \sigma_f(t,u)\,\mathrm{d}u$. For later use we will also consider a bond paying a continuous coupon of $k(t)$ up to time T and a lump sum payment of $k(T)$ at time T. The time t price of such a bond is

$$B_t^{\mathrm{cpn}} = \int_t^T k(s)P_t(s)\,\mathrm{d}s + k(T)P_t(T).$$

Applying a Leibnitz-type rule for stochastic processes (which in the specific context is formally stated and proved in the appendix which is available from the authors by request), it is seen that the coupon bond price must evolve according to

$$\mathrm{d}B_t^{\mathrm{cpn}} = -k(t)\,\mathrm{d}t + B_t^{\mathrm{cpn}}\left[(r_t + \sigma'_{B_t^{\mathrm{cpn}}}\lambda_B(t))\,\mathrm{d}t + \sigma'_{B_t^{\mathrm{cpn}}}\,\mathrm{d}w_{Bt}\right],$$

where

$$\sigma_{B_t^{\mathrm{cpn}}} = \frac{\int_t^T k(s)P_t(s)\sigma_P(t,s)\,\mathrm{d}s + k(T)P_t(T)\sigma_P(t,T)}{\int_t^T k(s)P_t(s)\,\mathrm{d}s + k(T)P_t(T)}. \tag{20}$$

Our specific results on how to hedge against changes in interest rates, as stated in Proposition 2, are based on the following assumption.

Assumption 1. The market price of risk process $\lambda_t \equiv \lambda(t)$ and the forward rate volatilities $\sigma_f(t,\tau)$ are deterministic functions of time.

The implications of the assumption that market prices of risk and forward rate volatilities are deterministic are important since we only allow interest rates to change stochastically and, hence, there are no reasons to hedge against stochastic changes in market prices of risk or forward rate volatilities. Also, as a consequence of Assumption 1 the following analysis is limited to Gaussian models of the term structure of interest rates. However, note that we do not assume that the diffusion coefficients σ_B, σ_{S1}, and σ_{S2} of the investment assets are deterministic and, in fact, they may be described by non-Markovian processes.

Multi-factor Gaussian models are in many respects flexible and thus often used for derivative pricing since they allow closed-form solution for most European-type term structure contingent claims (e.g., Amin and Jarrow, 1992; Brace and Musiela, 1994). A shortcoming of Gaussian term structure models, though, is that they are not able to rule out negative interest rates. The Gaussian assumption also allows closed-form expressions for optimal investment strategies, as we shall see in the following. Furthermore, it is important to point out that also in Gaussian HJM models, the short rate process is not necessarily Markovian (as is the case in the HJM three-factor example considered in a subsequent section).[8]

From the assumption that prices of risk and forward rate volatilities are deterministic, it follows that the short-term interest rate is normally distributed (Gaussian) and that the pricing kernel ζ_t, as stated in (3), is lognormally distributed. It is thus possible to compute in closed form the expectations in the definition of Q_t in (10) and, hence, obtain an analytical expression for Q_t. The proof of the following proposition is based on this feature.

Proposition 2. *Under Assumption 1, the value function and the optimal consumption strategy are given by* (12) *and* (13) *in Proposition 1, where in this case*

$$Q_t = \int_t^T Z_t(s)\,\mathrm{d}s + Z_t(T) \tag{21}$$

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[8] In fact, the short rate is only Markovian if $\sigma_f(t,\tau)$ can be separated as $\sigma_f(t,\tau) = G(t)H(\tau)$, where H is a real-valued continuously differentiable function that never changes sign and G is an $\mathbb{R}^k$-valued continuously differentiable function (cf. Carverhill, 1994).

with

$$Z_t(s) = K^{\frac{1}{\gamma}}(P_t(s))^{\frac{\gamma-1}{\gamma}} \exp\left\{ -\frac{\beta}{\gamma}(s-t) + \frac{1-\gamma}{2\gamma^2}g(t,s) \right\}, \quad 0 \leqslant t \leqslant s < T, \quad (22)$$

$$Z_t(T) = (1-K)^{\frac{1}{\gamma}}(P_t(T))^{\frac{\gamma-1}{\gamma}} \exp\left\{ -\frac{\beta}{\gamma}(T-t) + \frac{1-\gamma}{2\gamma^2}g(t,T) \right\} \quad (23)$$

and

$$g(t,s) = \int_t^s \|\lambda(u)\|^2 \, \mathrm{d}u + \int_t^s \|\sigma_P(u,s)\|^2 \, \mathrm{d}u - 2\int_t^s \lambda_B(u)'\sigma_P(u,s)\,\mathrm{d}u. \quad (24)$$

The optimal portfolio policy at time t is described by

$$\pi_t = \left(\frac{1}{\gamma}\right)(\sigma_t')^{-1}\lambda(t) + \left(\frac{\gamma-1}{\gamma}\right)(\sigma_t')^{-1}\left(\begin{array}{c}\sigma_{B_t^{\mathrm{cpn}}}\\0\end{array}\right), \quad (25)$$

where $\sigma_{B_t^{\mathrm{cpn}}}$ is the volatility vector of a bond, as defined in Eq. (20), which pays continuous coupon according to

$$k(s) = \widehat{E}_t^s[C_s]$$
$$= K^{\frac{1}{\gamma}}\frac{W_t}{Q_t}(P_t(s))^{-\frac{1}{\gamma}}\exp\left\{ -\frac{\beta}{\gamma}(s-t) + \frac{1-\gamma}{2\gamma^2}g(t,s) \right\}, \quad 0 \leqslant t \leqslant s < T, \quad (26)$$

and has a terminal lump sum payment at time T of

$$k(T) = \widehat{E}_t^T[W_T]$$
$$= (1-K)^{\frac{1}{\gamma}}\frac{W_t}{Q_t}(P_t(T))^{-\frac{1}{\gamma}}\exp\left\{ -\frac{\beta}{\gamma}(T-t) + \frac{1-\gamma}{2\gamma^2}g(t,T) \right\}. \quad (27)$$

Proof. The proof is available from the authors by request. □

Proposition 2 provides an explicit expression for the optimal investment strategy with possibly non-Markovian and multi-factor dynamics of interest rates. The optimal portfolio policy in (25) is described by two terms: the first term describes the usual speculative demand for risky assets while the second term describes the hedging demand for risky assets. The form of the hedging term is such that by choosing risky asset weights according to this term (and the residual invested in the risk-free bank account), one obtains a bond portfolio that basically replicates a specific coupon bond. This specific coupon bond will be referred to as the *hedge* bond in the following. In particular, Proposition 2 shows that with utility from intermediate consumption this hedge bond is equivalent to a coupon bond with coupon rates equal to the certainty equivalents of optimally planned future consumption rates.

In the special case of utility from terminal wealth only (corresponding to $K = 0$), the relevant bond for hedging reduces to a zero-coupon bond that expires at the investment horizon. This is similar to the insight obtained in specialized Vasicek settings by Brennan and Xia (2000) and Sørensen (1999). A zero-coupon bond seems an

intuitively appealing instrument for hedging changes in interest rates in the case of utility from terminal wealth only since this security has a certain payment at the investment horizon irrespectively of how interest rates evolve. Likewise, in the case including utility of intermediate consumption the suggested coupon bond seems a reasonable instrument for hedging shifts in interest rates in the sense that the certain payments on the bond match the currently planned future consumption expenditure profile irrespectively of how interest rates evolve.

The optimal portfolio policy described in (25) can in fact be implemented by allocating a fraction of wealth $(1/\gamma)$ into the speculative portfolio and a fraction of wealth $(1 - 1/\gamma)$ into the appropriate hedge bond. In order to see this, let $\bar{\pi}_t$ be the $\mathbb{R}^{d+1}$-valued vector process describing the *augmented* optimal portfolio weights where the fraction of wealth invested in the risky assets, π_t, are included as the first d-entries while the fraction of wealth invested in the risk-free bank account is included as the $(d+1)$th entry. Note that by inserting the optimal risky asset portfolio weights in (25), the optimal augmented portfolio policy can be obtained in the form

$$
\bar{\pi}_t = \begin{pmatrix} \pi_t \\ 1 - 1_d' \pi_t \end{pmatrix}
$$

$$
= \left(\frac{1}{\gamma} \right) \begin{pmatrix} (\sigma_t')^{-1} \lambda(t) \\ 1 - 1_d' (\sigma_t')^{-1} \lambda(t) \end{pmatrix} + \left(1 - \frac{1}{\gamma} \right) \begin{pmatrix} (\sigma_t')^{-1} \begin{pmatrix} \sigma_{B_t^{\mathrm{cpn}}} \\ 0 \end{pmatrix} \\ 1 - 1_d' (\sigma_t')^{-1} \begin{pmatrix} \sigma_{B_t^{\mathrm{cpn}}} \\ 0 \end{pmatrix} \end{pmatrix}. \tag{28}
$$

The first term in (28) describes the augmented optimal portfolio weights in the log-utility case where $\gamma = 1$. This portfolio is usually referred to as the growth-optimal portfolio or, equivalently, the speculative portfolio. On the other hand, the last term in (28) describes the augmented portfolio weights needed to implement the appropriate hedge bond.

According to Proposition 2, the specific dynamics of the term structure of interest rates is of importance for how to hedge against changes in the opportunity set only through its effect on the optimal forward-expected consumption pattern. In the following examples, we will focus on the determinants of the optimal forward-expected consumption patterns and, in particular, our focus is on whether the current form of the term structure or the dynamics of the term structure is of crucial importance for the optimal forward-expected consumption pattern. In this context it can be noted that, even in the general setting of Section 2, only the form of the term structure matters for the optimal forward-expected consumption patterns for the benchmark cases of log-investors and infinitely risk averse investors, while the term structure dynamics is irrelevant. For infinite risk aversion, this follows from the fact that the optimal consumption rate is constant and equal to W_t/Q_t where Q_t describes the price of an annuity bond which is fully determined by the prevailing term structure at time t, cf. the description of Q_t in (15) for this special case. For log utility, it can be shown that the forward-expected optimal consumption rate is

$$
\widehat{E}_t^s[C_s] = W_t \frac{K}{A(t)} (P_t(s))^{-1} e^{-\beta(s-t)}, \quad 0 \leqslant t \leqslant s < T, \tag{29}
$$

with $A(t)$ being defined in Proposition 1. The forward-expected terminal wealth at time T is given by a similar expression which is also fully determined by the current term structure of interest rates, as reflected in zero-coupon bond prices $P_t(s)$, and not influenced by the parameters describing the term structure dynamics.

4. Specific examples

In this section we consider two specific examples of interest rate dynamics in the set-up of the previous section. The first example is based on the term structure dynamics from the Vasicek (1977) model while the second example is based on a flexible three-factor HJM term structure model where the term structure can exhibit three different kinds of changes: a parallel level change, a slope change, or a curvature change. As shown by Heath et al. (1992), any dynamic interest rate model is fully specified by the current form of the term structure and the forward rate volatilities. Hence, the HJM framework is natural for the purpose of comparing the separate effects of the current *form* of the term structure and the *dynamics* of the term structure on the optimal interest rate hedging strategy. In our specific examples the initial term structures are thus chosen to be identical across the two examples, i.e. the initial form of the term structure curve in the three-factor HJM term structure model is adopted from the Vasicek example. We compute the optimal strategies in both examples using empirically reasonable parameter values. For various degrees of risk aversion and for different forms of the initial term structure we compare the relevant hedge bond under Vasicek dynamics and the relevant hedge bond under the three-factor HJM dynamics. Our results below suggest that the optimal payment schedule on the hedge bond is very sensitive to the form of the initial term structure of interest rates while the optimal payment schedule on the hedge bond is insensitive to the dynamics of interest rates over time when the current term structure is held fixed.

4.1. Vasicek example

In the following example we allow for utility from both intermediate consumption and terminal wealth by setting the preference parameter K equal to $\frac{1}{2}$ in the specification of the utility function in (1). This implies that utility from intermediate consumption and utility from terminal wealth are weighted equally. The set-up for investment assets in the following example is basically as in Brennan and Xia (2000) and Sørensen (1999), but they only consider utility from terminal wealth. The agent can invest in a single stock and a single bond as well as the "instantaneously" risk-free bank account. The term structure dynamics is described by the one-factor term structure model originally suggested by Vasicek (1977). In particular, the dynamics of the short-term risk-free interest rate is described by an Ornstein–Uhlenbeck process of the form

$$\mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t - \sigma_r\,\mathrm{d}w_{Bt}, \qquad (30)$$

where the parameter θ describes the long-run level for the short-term interest rate, κ is a mean-reversion parameter that determines the strength of tendency to the long-run level, and the parameter σ_r describes the interest rate volatility. Besides the parameters describing the interest rate dynamics, the parameter denoted λ_B in the context of Section 3 determines the price of interest rate risk.

Using standard no-arbitrage arguments, one can solve for prices on interest rate contingent claims in the Vasicek model. The possible forms of the term structure of forward interest rates can thus be determined by solving for prices on zero-coupon bonds. The τ-maturity forward rate at time t in the Vasicek model is given by

$$f_t(\tau) = \mathrm{e}^{-\kappa(\tau-t)}r_t + r_\infty(1 - \mathrm{e}^{-\kappa(\tau-t)}) + \frac{\sigma_r^2}{2\kappa}\mathrm{e}^{-\kappa(\tau-t)}b(\tau - t), \qquad (31)$$

where

$$r_\infty = \theta + \frac{\lambda_B \sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2},$$

$$b(s) = \frac{1}{\kappa}(1 - \mathrm{e}^{-\kappa s}).$$

The dynamics of the τ-maturity forward rate can be determined from (31) and (30) and an application of Ito's lemma. In particular, it is seen that the forward rate volatility structure in this example has the form $\sigma_f(t, \tau) = -\sigma_r \mathrm{e}^{-\kappa(\tau-t)}$. Within the HJM framework of Section 3, this volatility structure and an initial term structure of forward rates of the form in (31) provide a complete specification of the Vasicek (1977) term structure model.

The agent can invest in a single stock as well as bonds and the bank account. In the specific case of a one-factor interest rate model it is sufficient that the agent can invest in a single bond besides the stock and the bank account in order to implement the complete-market optimal solution. The price process of the single stock is described in Eq. (17) where in this case σ_{S1} and σ_{S2} are scalars (i.e. of dimension 1×1).

The specific parameter values used in the following numerical example are chosen as follows:

$$\theta = 0.04, \quad \kappa = 0.15, \quad \sigma_r = 0.015,$$
$$\sigma_{S1} = 0.0625, \quad \sigma_{S2} = 0.2421, \qquad (32)$$
$$\varphi_S = 0.05, \quad \lambda_S = 0.19365, \quad \lambda_B = 0.05.$$

In particular, the parameters κ, θ, and σ_r, which describe the interest rate dynamics, are chosen so that they are close to those obtained by Chan et al. (1992) for the Vasicek interest rate process. The parameters for the stock process are chosen so that the expected excess return on the stock is $\varphi_S = 5\%$, the volatility of the stock is constant 25% $(= (\sigma_{S1}^2 + \sigma_{S2}^2)^{1/2})$, and the "instantaneous" correlation coefficient between the stock and the short-term interest rate is constant -25% (and, hence, the correlation between the stock and any bond in the one-factor Vasicek model is 25%). The 5% expected rate of excess return on the stock is below the 8.4% point estimate suggested by the Ibbotson Associates 1926–1994 historical returns data on stocks

(see, e.g., Brealey and Myers, 1996, Chapter 7, Table 7-1). Though, as pointed out by Brown et al. (1995), the use of realized mean returns in this context is likely to involve a survival bias which could be as high as 400 basis points per year.

The 25% volatility of the stock is slightly higher than the 20.2% historical volatility estimate on the S&P 500 index based on the Ibbotson Associates returns data (see, e.g., Brealey and Myers, 1996, Chapter 7) but well in accordance with, say, volatilities on individual stocks and less diversified portfolios of stocks. Furthermore, the 25% positive correlation between the stock and bonds is consistent with the empirical results in e.g. Campbell (1987), Fama and French (1989), and Shiller and Beltratti (1992). Finally, the risk premia on bonds, $\lambda_B = 0.05$, implies that e.g. the expected excess return on a 10-year zero-coupon bond in the Vasicek model is 0.39%. [9]

The above parameter values imply that an agent with logarithmic utility invests an 80% fraction of wealth in the stock, a fraction of 0% in bonds, and the residual 20% of wealth in the bank account. Hence, the speculative portfolio under the specific parameter values involve no speculative demand for bonds. Agents with non-logarithmic utility, however, want to invest in a bond, or bond portfolio, that has payoffs that equal their forward-expected consumption pattern in order to hedge against changes in the opportunity set, as described in Proposition 2. In line with the discussion after Proposition 2, the appropriate investor specific bond in this respect is referred to as the hedge bond. As formalized in (28), the infinitely risk averse investors invest 100% in the hedge bond while e.g. an investor with constant relative risk aversion, γ, equal to 2 will invest 50% $(= 1/\gamma)$ of wealth in the speculative portfolio and 50% $(= 1 - 1/\gamma)$ of wealth in the hedge bond; i.e. the portfolio composition in this case is: 40% in the stock, 10% in the bank account, and 50% in the hedge bond. The optimal asset allocations of investors with risk aversion parameters 1, 4/3, 2, 4, and infinity are tabulated in Table 1 in accordance with (28).

It can be noted that the asset allocations tabulated in Table 1 do not depend on the time horizons of the investors; however, the appropriate hedge bonds differ across investors that are heterogeneous with respect to both risk aversion and time horizon. Moreover, the asset allocation choices in Table 1 do not depend on the form of the current term structure, but the optimal payment schedules on the hedge bonds do. Finally, it may be noted that if the relevant coupon bonds for hedging are not explicitly available in the market, they can always be replicated by trading in any single bond and the bank account within the Vasicek model.

We will consider the optimal payment schedules on the relevant hedge bonds in three cases with different initial term structures of forward rates. These three forms are given by setting the short-term interest rate equal to 0.01, 0.04, and 0.07, respectively. The three forms of the initial term structure of forward interest rates are displayed in Fig. 1.

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[9] Again, Brealey and Myers (1996, Chapter 7) tabulate the average historical excess return on government bonds to be slightly higher, 1.4%, based on the Ibbotson Associates (1995) returns data.

Table 1
Optimal asset allocations for investors with different degrees of relative risk aversion

	Relative risk aversion				
	$\gamma = 1$	$\gamma = 4/3$	$\gamma = 2$	$\gamma = 4$	$\gamma = \infty$
Stock	80%	60%	40%	20%	0%
Bank account	20%	15%	10%	5%	0%
Hedge bond	0%	25%	50%	75%	100%

The optimal allocations are in accordance with (28), and relevant Vasicek model parameter values and stock price parameter values are given in (32).

Note: The optimal asset allocations are identical for investors with different time horizons (T) and time preference parameters (β). The appropriate hedge bonds, however, differ across investor types. Also, the relevant hedge bonds for the different investors in this table, which are heterogeneous with respect to degree of relative risk aversion, are not identical.
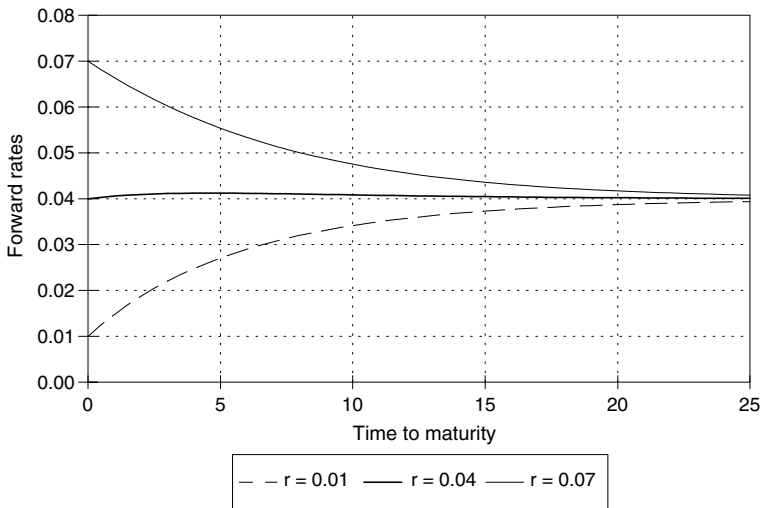

Fig. 1. *Term structures of forward interest rates.* The figure displays forward rates as a function of time to maturity for different Vasicek term structures described by short interest rate levels of 0.01, 0.04, and 0.07, respectively.

As formalized in Proposition 2, the forward-expected consumption pattern of the agent is crucial for how the agent should optimally hedge against changes in interest rates. The forward-expected consumption pattern and the forward-expected terminal wealth of the agent are described by the expressions in (26) and (27). In particular, the consumption pattern over time depends on the term structure of forward rates through the occurrence of the zero-coupon price $P(t, \tau) = \exp(-\int_t^\tau f(t, s)\, \mathrm{d}s)$ in the expressions. Also, the consumption pattern over time depends on the prices on risk in the economy through the expression for the variance of the log-pricing kernel, $g(t, s)$ as stated in Eq. (24). Using that the zero-coupon bond volatility is

$\sigma_P(t, \tau) = - \int_t^\tau \sigma_f(t, u) \, du = \sigma_r b(\tau - t)$ and by evaluating the integrals in (24), one obtains

$$g(t, s) = (\lambda_B^2 + \lambda_S^2)(s - t) + 2(r_\infty - \theta)(b(s - t) - (s - t)) - \frac{\sigma_r^2}{2\kappa}(b(s - t))^2. \quad (33)$$

The forward-expected consumption patterns are displayed in Fig. 2 for different degrees of relative risk aversion, a subjective time discount rate of $\beta = 0.03$, and a time horizon of $T = 25$ (years). The investors have initial wealth of $W_0 = 100$.

The consumption patterns in the figure describe the specific payment schedules for the relevant coupon bonds that the different investors should use in order to hedge against changes in the term structure of interest rates. The log-utility investors and the infinitely risk averse investors are polar benchmark cases where either the demand for the hedge bond is exactly 0% or exactly 100%. Investors in between these two polar cases will invest a fraction of wealth between 0% and 100% in the specific
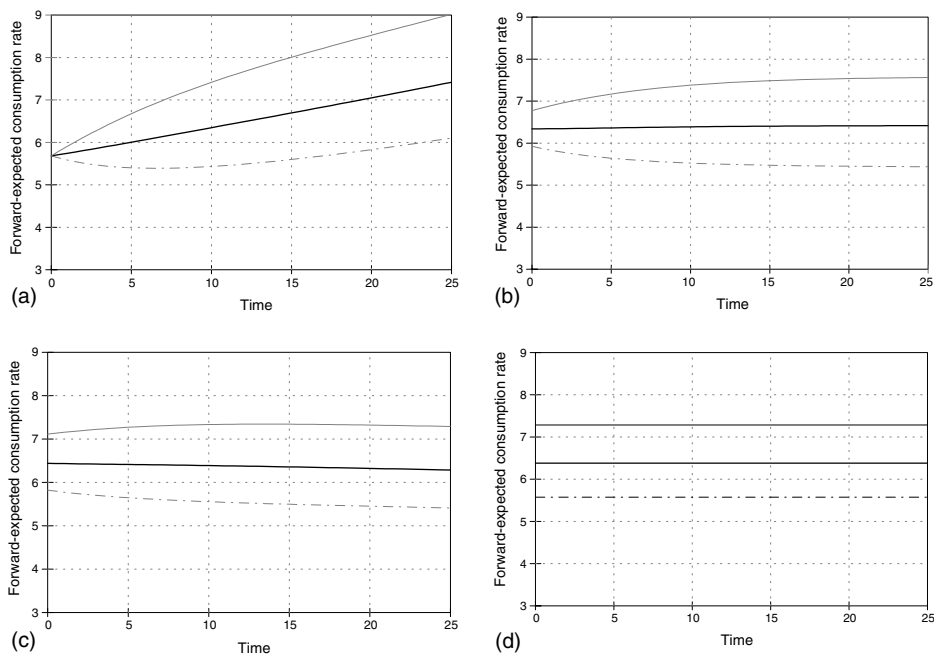

Fig. 2. Expected consumption patterns with initial wealth $W_0 = 100$ and time horizon $T = 25$. The figure displays the forward-expected consumption streams for four different levels of the relative risk aversion coefficient γ; panel (a): $\gamma = 1$, panel (b): $\gamma = 2$, panel (c): $\gamma = 4$, panel (d): $\gamma = \infty$. The three curves in each panel correspond to the different initial forward rate curves displayed in Fig. 1. The dashed curve is for the upward sloping term structure ($r = 0.01$), the thick solid curve is for the nearly flat term structure ($r = 0.04$), and the thin solid curve is for the downward sloping term structure ($r = 0.07$). The present value of the consumption policy must equal current wealth, and the discounted value of the forward-expected consumption stream is thus in all cases $W_0 = 100$. Moreover, the current consumption-to-wealth ratios in percent are described by the current, time $t = 0$, consumption rates.

bonds in order to hedge against changes in the opportunity set. For example, the illustrated investors in Fig. 2(b) and (c) with relative risk aversion of 2 and 4 should optimally invest 50% and 75% in their specific hedge bonds, cf. Table 1.

For a log-utility investor and for an investor with $\gamma = \infty$, the forward-expected consumption and terminal wealth patterns only depend on the initial term structure of interest rates, as described in the discussion following Proposition 2. In particular, for $\gamma = \infty$ the forward-expected consumption pattern is always flat, as displayed in Fig. 2(d), while the forward-expected consumption pattern for a log-utility investor in Fig. 2(a) depends on the subjective discount rate β and the specific form of the current term structure. From (29) it follows that in the logarithmic utility case, $\gamma = 1$, the forward-expected consumption rate $k(s)$ must satisfy

$$k'(s) = (f_t(s) - \beta)k(s)$$

and, hence, that the forward-expected consumption rate as a function of the time to consumption is increasing whenever the forward rate is higher than the subjective discount rate $\beta = 0.03$, and vice versa. Furthermore, the consumption-to-wealth ratios are described by the current (time $t = 0$) consumption rates, and according to Fig. 2(a) the current consumption-to-wealth ratios are identical across the three term structure cases for the log-utility investors with $C_0/W_0 = 5.686/100 = 5.686\%$. On the other hand, the optimal constant consumption rates that can be sustained by the infinitely risk-averse investors are determined entirely by the current annuity bond price which differs across the three term structure cases. The consumption patterns for the investors in Fig. 2(b) and (c) are basically in between the two polar benchmark cases of investors with logarithmic utility and infinitely risk-averse investors.

4.2. A non-Markovian three-factor HJM model

This example features non-Markovian dynamics of the opportunity set. We consider three different initial term structures of forward rates as input in the HJM modeling approach. The relevant current term structures are adopted from the above Vasicek example, as displayed in Fig. 1; the entire term structures of forward rates in Fig. 1 are thus used as an input in the investment/consumption decision problem.

The term structure can basically exhibit three kinds of changes: a parallel level change, a slope change, and a curvature change. Specifically, the forward rate volatility structure is assumed to have the form

$$\sigma_f(t, \tau)' = -\big(\sigma_1, \sigma_2 e^{-\kappa_2(\tau-t)}, \sigma_3(\tau - t)e^{-\kappa_3(\tau-t)}\big), \quad 0 \leqslant t \leqslant \tau \leqslant T. \tag{34}$$

The dynamics of the forward rate curve is described by inserting the volatility structure (34) in (18). In particular, a change in the Wiener process that governs movements in the first factor will result in an equal change in all forward rates for different maturities; hence, this causes a parallel level change of the forward curve. Likewise, a change in the Wiener process that governs movements in the second factor will significantly affect forward rates with short maturities but not forward rates with long maturities, and this thus causes a slope change of the forward curve.

Finally, a change in the Wiener process that governs movements in the third factor will affect forward rates with medium maturities but neither forward rates with short and long maturities, and this causes a change in the curvature of the forward curve. The three factors are similar to the fundamental three components in the Nelson and Siegel (1987) structural forms widely used in practice for calibration of term structures of interest rates and also consistent with the term structure factors determined empirically by e.g. Litterman and Scheinkman (1991).

The volatility of any zero-coupon bond is described by $\sigma_P(t, \tau) = - \int_t^\tau \sigma_f(t, u) \, \mathrm{d}u$ and under the above specification of forward curve volatility we have

$$\sigma_P(t, \tau)' = \left(\sigma_1(\tau - t), \sigma_2 b_2(\tau - t), \frac{\sigma_3}{\kappa_3} (b_3(\tau - t) - (\tau - t)\mathrm{e}^{-\kappa_3(\tau - t)}) \right), \tag{35}$$

where $b_j(\tau) = \frac{1}{\kappa_j}(1 - \mathrm{e}^{-\kappa_j \tau})$ for $j = 2, 3$.

As in the Vasicek example above, it is possible to determine the optimal forward-expected consumption pattern and, hence, the relevant coupon bond for hedging against changes in the opportunity set using the general results in Proposition 2. Besides the form of the initial term structure of interest rates the variance of the (log) pricing kernel, $g(t, s)$, determines the relevant consumption patterns in (26) and (27). Straightforward calculations using (24) show that the analogy to (33) in the Vasicek example is now given by

$$
\begin{aligned}
g(t, s) = {} & (\lambda_{B1}^2 + \lambda_{B2}^2 + \lambda_{B3}^2 + \lambda_S^2)(s - t) - \lambda_{B1}\sigma_1(s - t)^2 + \frac{1}{3}\sigma_1^2(s - t)^3 \\
& + \left(\frac{2\lambda_{B2}\sigma_2}{\kappa_2} - \frac{\sigma_2^2}{\kappa_2^2} \right)(b_2(s - t) - (s - t)) - \frac{\sigma_2^2}{2\kappa_2}(b_2(s - t))^2 \\
& - \left(\frac{4\lambda_{B3}\sigma_3}{\kappa_3^2} - \frac{3}{2}\frac{\sigma_3^2}{\kappa_3^4} \right)(s - t) - \frac{1}{2}\frac{\sigma_3^2}{\kappa_3^3}(s - t)^2 \\
& + \left(\frac{4\lambda_{B3}\sigma_3}{\kappa_3^2} - \frac{3}{2}\frac{\sigma_3^2}{\kappa_3^4} + \left(\frac{2\lambda_{B3}\sigma_3}{\kappa_3} + \frac{\sigma_3^2}{\kappa_3^3} \right)(s - t) + \frac{\sigma_3^2}{\kappa_3^2}(s - t)^2 \right) b_3(s - t) \\
& - \frac{\sigma_3^2}{\kappa_3^2} \left(\frac{5}{4\kappa_3} + \frac{3}{2}(s - t) + \frac{1}{2}\kappa_3(s - t)^2 \right)(b_3(s - t))^2. \tag{36}
\end{aligned}
$$

In the following, we will tabulate numerical results for three different sets of parameters for the three-factor HJM model. Our base case set of parameters is chosen such that the volatilities of short term and long term bonds as well as the expected excess returns on stocks and bonds are of the same magnitude as in the Vasicek example above. Below, we will comment further on how this is achieved but, specifically, the parameter values in the base case are:

$$\kappa_2 = 1.00, \quad \kappa_3 = 0.50, \quad \sigma_1 = 0.00325, \quad \sigma_2 = 0.01184, \quad \sigma_3 = 0.00869,$$
$$\sigma_{S1} = (0.03187, 0.02305, 0.04857)', \quad \sigma_{S2} = 0.24206,$$
$$\varphi_S = 0.05, \quad \lambda_S = 0.19365, \quad \lambda_B = (0.02549, 0.01844, 0.03886)'. \tag{37}$$

In choosing the parameters in (37) we first fixed κ_2 and κ_3, which determine the slope effect and the curvature effect in the dynamics of the forward rate curve in (18). Our rationale for choosing the specific parameter values is given below, but the following numerical results are not sensitive to the specific parameter values used for κ_2 and κ_3. [10] In the present context, the innovations in the forward curve are generated by a three-dimensional Wiener process, $w_B = (w_{B1}, w_{B2}, w_{B3})'$. As described above, an innovation in w_{B1} affects all forward rates equally while e.g. an innovation in w_{B2} affects short rates but not very long rates. For example, $\kappa_2 = 1.00$ implies that if an innovation in w_{B2} increases the spot rate with 100 basis point, the 1-year forward rate is only increased by $(100 \times e^{-\kappa_2 \times 1} =)$ 36.79 basis points, and the 5-year forward rate is only increased by 0.67 basis points; hence, an innovation in w_{B2} will significantly change the slope of the forward rate curve. Likewise, an innovation in w_{B3} will not affect the very near forward rates nor the very distant forward rates but will change the curvature of the forward rate curve. The maximum amplitude in the forward rate curve caused by an innovation in w_{B3} occurs for a medium distant forward rate; specifically, for $\kappa_3 = 0.50$ the maximum amplitude occurs for the $(1/\kappa_3 =)$ 2-year forward rate. Hence, the specific parameter values chosen for κ_2 and κ_3 are reasonable in order to empirically capture what is usually referred to as a slope change and a curvature change in the term structure, and this is the rationale for the specific parameter choices.

While the parameters κ_2 and κ_3 are specified exogenously, the forward rate volatility parameters σ_1, σ_2, and σ_3 are calibrated in order to ensure that the volatilities of zero-coupon bonds with times to maturity equal to 0.25, 2, and 10 years, respectively, are identical to those in the Vasicek example. [11] Next, σ_{S1} and σ_{S2} are chosen so that the volatility on the stock is 25% and so that the correlation coefficients between the stock and any of the three term structure factors are −25%, which corresponds to the −25% correlation between the stock and the short-term interest rate in the Vasicek example. Finally, risk premia are also calibrated to be comparable to those in the Vasicek example. In particular, the expected excess return on the stock is 5% while the risk premia on bonds, as reflected in λ_B, are calibrated so that there is no speculative demand for bonds (also, $\|\lambda_B\| = 0.05$, as in the Vasicek example). [12] The portfolio choice of a logarithmic investor is, hence, to invest 80% of wealth in the stock, 0% in bonds, and 20% in the bank account, as in the Vasicek example. Likewise, other investors allocate the same fraction of wealth into the stock, the bank account, and a hedge bond, as in the Vasicek example; cf. Table 1.

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[10] Numerical results based on alternative parameter values for κ_2 and κ_3 (varied separately in intervals from 0.10 to 10) are available from the authors by request.

[11] This is done by equating the relevant zero-coupon bond volatilities in (35) to those in the Vasicek example, and thus basically solving three equations with respect to the three unknowns: σ_1, σ_2, and σ_3.

[12] This is achieved by choosing the three parameters in λ_B so that there is no speculative demand for bonds exposed alone to innovations in w_{B1}, w_{B2}, and w_{B3}, respectively. In practice, the speculative demand for these bonds, as described by the first term in (25), are equated to zero, and the three parameters in λ_B are thus basically obtained by solving three equations with respect to these three parameters.

The speculative demand for securities in this example is by construction exactly similar to the speculative demand in the above Vasicek example. The way the investors want to hedge against changes in the opportunity set, however, may be quite different compared to the Vasicek case due to the more complex dynamics of the term structure of interest rates in this HJM three-factor setting. In our view, a comparison between the hedge choice in the Vasicek example and in this HJM three-factor setting using the base case parameters in (37) is relevant for addressing questions such as: (i) is the present form of the term structure of interest rates important for how to hedge against changes in the opportunity set? and (ii) is the flexibility and dynamics of the term structure of interest rates important for how to hedge against changes in the opportunity set when the current term structure is kept fixed?

As formalized in Proposition 2 the forward-expected consumption pattern is crucial for the hedging behavior since the appropriate bond (or bond portfolio) for hedging against changes in the opportunity set is one that has a payment schedule similar to the optimal forward-expected consumption pattern. Hence, the questions above can be answered by comparing the optimal consumption patterns across different scenarios. The first question, (i), can be addressed by looking at the diversity of consumption patterns under different current term structures, but under the same fundamental model of interest rate dynamics. (Such results have already been presented in the above numerical analysis under Vasicek interest rate dynamics.) The second question, (ii), can be addressed by looking at the diversity of consumption patterns under different term structure dynamics, but by using the same current term structure as input in the analysis. This is done below where we make numerical comparisons across the Vasicek model and the HJM three-factor model when the same current term structure of forward rates applies; in particular, this analysis is based on using the entire Vasicek term structures of forward rates exhibited in Fig. 1 as input current term structures in the HJM three-factor model.

The optimal consumption patterns are tabulated in Table 2 for investors with degrees of relative risk aversion equal to 1, 4/3, 2, 4, and infinity so that the different investors invest 0%, 25%, 50%, 75%, and 100%, respectively, in their appropriate hedge bonds; cf. Table 1. As in the Vasicek example, the investors have an investment horizon of 25 years, a subjective time discount rate of $\beta = 0.03$, and they equally weight utility from intermediate consumption and final wealth, i.e. $K = \frac{1}{2}$ in the general utility function specification in (1). The investors have initial wealth $W_0 = 100$.

The forward-expected consumption patterns for the Vasicek dynamics are exactly identical to those displayed in Fig. 2 in the Vasicek example above. The forward-expected consumption patterns for the HJM three-factor model are for the benchmark parameters in (37) and by using the Vasicek forward rate term structures in Fig. 1 as separate current term structure inputs. [13] Also, as discussed in relation to

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[13] The relevant Vasicek input forward rate term structures are given in analytical form by (31). The tabulated forward rates in Table 2 basically represent single points on the entire term structure of forward rates which constitutes the input to the analysis.

Table 2
Forward-expected consumption rates (i.e. payment schedules for the relevant coupon bonds to hedge changes in the opportunity set) for investors with initial wealth $W_0 = 100$, $T = 25$, $\beta = 0.03$, $K = 1/2$, and different degrees of relative risk aversion

Time	Forward rate	$\gamma = 1$		$\gamma = 4/3$		$\gamma = 2$		$\gamma = 4$		$\gamma = \infty$	
		Vasicek	HJM	Vasicek	HJM	Vasicek	HJM	Vasicek	HJM	Vasicek	HJM
Vasicek term structure of forward rates with short interest rate, r, equal to 0.01											
0.00	0.0100	5.686	5.686	5.883	5.883	5.929	5.928	5.824	5.824	5.576	5.576
5.00	0.0271	5.404	5.404	5.562	5.561	5.644	5.641	5.649	5.647	5.576	5.576
10.00	0.0342	5.434	5.434	5.487	5.486	5.527	5.526	5.557	5.556	5.576	5.576
15.00	0.0373	5.598	5.598	5.510	5.511	5.476	5.477	5.498	5.500	5.576	5.576
20.00	0.0387	5.830	5.830	5.577	5.580	5.453	5.456	5.453	5.455	5.576	5.576
25.00	0.0394	6.102	6.102	5.665	5.667	5.442	5.445	5.413	5.416	5.576	5.576
Vasicek term structure of forward rates with short interest rate, r, equal to 0.04											
0.00	0.0400	5.686	5.686	6.086	6.086	6.342	6.341	6.442	6.442	6.384	6.384
5.00	0.0412	6.005	6.005	6.228	6.226	6.364	6.361	6.415	6.413	6.384	6.384
10.00	0.0409	6.347	6.347	6.378	6.377	6.390	6.388	6.390	6.389	6.384	6.384
15.00	0.0405	6.695	6.695	6.519	6.520	6.406	6.407	6.360	6.361	6.384	6.384
20.00	0.0402	7.050	7.050	6.653	6.657	6.414	6.418	6.325	6.328	6.384	6.384
25.00	0.0401	7.418	7.418	6.784	6.787	6.418	6.422	6.288	6.291	6.384	6.384
Vasicek term structure of forward rates with short interest rate, r, equal to 0.07											
0.00	0.0700	5.686	5.686	6.295	6.294	6.778	6.778	7.114	7.113	7.285	7.285
5.00	0.0554	6.673	6.673	6.972	6.970	7.170	7.167	7.273	7.271	7.285	7.285
10.00	0.0476	7.414	7.414	7.413	7.411	7.381	7.379	7.336	7.334	7.285	7.285
15.00	0.0434	8.006	8.006	7.711	7.712	7.487	7.489	7.344	7.345	7.285	7.285
20.00	0.0417	8.525	8.525	7.936	7.940	7.539	7.544	7.324	7.328	7.285	7.285
25.00	0.0408	9.017	9.017	8.124	8.128	7.563	7.568	7.290	7.294	7.285	7.285

Initial term structures of forward rates are from the Vasicek model with $r = 1\%$, 4%, and 7%. Payment schedules are tabulated for both the Vasicek and the HJM three-factor term structure dynamics. Relevant parameter values are given in (32) and (37), respectively. The current consumption to wealth ratios in percent are described by the time $t = 0$ consumption rates.

Fig. 2, the present value of the optimal consumption streams equal $W_0 = 100$ for all investors represented in the table. Furthermore, the consumption-to-wealth ratios are described by the current, time $t = 0$, consumption rates; e.g. $C_0/W_0 = 5.686\%$ for a log-utility investor in all term structure cases. As noted earlier, the results for log-utility investors and infinitely risk-averse investors in Table 2 are exactly identical since the forward-expected consumption patterns of these investors depend only on the current form of the term structure of interest rates. However, also for investors with relative risk aversion in between these benchmark investors the differences between the consumption patterns in the Vasicek example and in the base case HJM three-factor model are small. A conclusion that can be drawn from observing similar forward-expected consumption patterns from the Vasicek example and the base case HJM three-factor model is that investors need not care about the dynamics of the term structure of interest rates since in both cases the investors should hedge against changes in the investment opportunity set by basically buying the same coupon bond. [14] On the other hand, the current form of the term structure is important for the optimal consumption patterns of the investors and is, hence, important for the precise payment schedule of the relevant bond for hedging against changes in the opportunity set as reflected in the clear diversity in consumption patterns across different current term structure cases in Table 2.

In Table 3 we have tabulated results for two other sets of parameter values for the HJM three-factor model.

In the discussion of Proposition 1, it was noted that optimal consumption choices are only altered if one changes the parameters that enter the dynamics (or particular moments) of the pricing kernel process. Therefore, changing e.g. the volatilities σ_{S1} and σ_{S2} of the investment assets will have no consequences for the optimal forward-expected consumption pattern and, hence, no consequences for the relevant coupon bond to hedge against changes in the opportunity set. On the other hand, if one changes risk premia or parameter values in the description of the term structure dynamics, the optimal consumption pattern will in general be affected. Therefore, we only consider two other sets of parameters: one in which forward-rate volatility parameters are changed and one in which risk premia parameters are changed.

The two sets of alternative parameters considered in Table 3 are:

$$\kappa_2 = 1.00, \quad \kappa_3 = 0.50, \quad \sigma_1 = 0.00650, \quad \sigma_2 = 0.02367, \quad \sigma_3 = 0.01738,$$
$$\sigma_{S1} = (0.03187, 0.02305, 0.04857)', \quad \sigma_{S2} = 0.24206,$$
$$\varphi_S = 0.05, \quad \lambda_S = 0.19365, \quad \lambda_B = (0.02549, 0.01844, 0.03886)' \tag{38}$$

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[14] If the relevant hedge bonds are not explicitly available in the market, they must be replicated. As a qualifier to this conclusion, it may then be noted that the replication strategies vary across the considered models. For example, in the case of the Vasicek model the investor can replicate the hedge bond by trading in the bank account and in a single bond. In the HJM model, three bonds are required.

Table 3
Forward-expected consumption rates (i.e. payment schedules for the relevant coupon bonds to hedge changes in the opportunity set) for investors with initial wealth $W_0 = 100$, $T = 25$, $\beta = 0.03$, $K = 1/2$, and different degrees of relative risk aversion

Time	Forward rate	$\gamma = 4/3$			$\gamma = 2$			$\gamma = 4$		
		HJM-1	HJM-2	HJM-3	HJM-1	HJM-2	HJM-3	HJM-1	HJM-2	HJM-3
Vasicek term structure of forward rates with short interest rate, r, equal to 0.01										
0.00	0.0100	5.883	5.894	5.669	5.928	5.943	5.645	5.824	5.834	5.614
5.00	0.0271	5.561	5.572	5.456	5.641	5.657	5.502	5.647	5.658	5.542
10.00	0.0342	5.486	5.495	5.475	5.526	5.537	5.512	5.556	5.565	5.546
15.00	0.0373	5.511	5.511	5.590	5.477	5.476	5.584	5.500	5.498	5.579
20.00	0.0387	5.580	5.559	5.751	5.456	5.429	5.682	5.455	5.435	5.624
25.00	0.0394	5.667	5.611	5.932	5.445	5.372	5.789	5.416	5.362	5.671
Fisher–Weil duration:		10.73	10.71	10.89	10.62	10.60	10.84	10.64	10.62	10.80
Vasicek term structure of forward rates with short interest rate, r, equal to 0.04										
0.00	0.0400	6.086	6.097	5.867	6.341	6.356	6.043	6.442	6.453	6.216
5.00	0.0412	6.226	6.239	6.111	6.361	6.378	6.209	6.413	6.426	6.300
10.00	0.0409	6.377	6.387	6.365	6.388	6.401	6.377	6.389	6.399	6.383
15.00	0.0405	6.520	6.519	6.615	6.407	6.405	6.537	6.361	6.359	6.459
20.00	0.0402	6.657	6.631	6.862	6.418	6.385	6.689	6.328	6.304	6.529
25.00	0.0401	6.787	6.720	7.107	6.422	6.336	6.832	6.291	6.227	6.592
Fisher–Weil duration:		10.64	10.62	10.80	10.44	10.42	10.66	10.37	10.35	10.53
Vasicek term structure of forward rates with short interest rate, r, equal to 0.07										
0.00	0.0700	6.294	6.306	6.070	6.778	6.793	6.464	7.113	7.125	6.870
5.00	0.0554	6.970	6.985	6.843	7.167	7.186	7.002	7.271	7.285	7.149
10.00	0.0476	7.411	7.423	7.399	7.379	7.394	7.372	7.334	7.345	7.334
15.00	0.0434	7.712	7.711	7.827	7.489	7.486	7.647	7.345	7.343	7.466
20.00	0.0417	7.940	7.909	8.187	7.544	7.505	7.868	7.328	7.299	7.568
25.00	0.0408	8.128	8.046	8.512	7.568	7.466	8.058	7.294	7.220	7.650
Fisher–Weil duration:		10.55	10.53	10.71	10.26	10.24	10.48	10.10	10.08	10.26

Initial term structures of forward rates are from the Vasicek model with $r = 1\%$, 4%, and 7% (as displayed in Fig. 1). Payment schedules are tabulated for the HJM three-factor term structure dynamics and for the three different sets of parameter values given in (37)–(39). The current consumption to wealth ratios in percent are described by the time $t = 0$ consumption rates.

and

$$\kappa_2 = 1.00, \quad \kappa_3 = 0.50, \quad \sigma_1 = 0.00325, \quad \sigma_2 = 0.01184, \quad \sigma_3 = 0.00869,$$
$$\sigma_{S1} = (0.03187, 0.02305, 0.04857)', \quad \sigma_{S2} = 0.24206,$$
$$\varphi_S = 0, \quad \lambda_S = 0, \quad \lambda_B = (0, 0, 0)', \tag{39}$$

respectively.

The parameter set in (38) differs from the base set of parameters in (37) alone by higher volatilities on the forward rate curve; specifically, the parameters in (38) are chosen so that the volatilities on zero-coupon bonds with time to maturity equal to 0.25, 2, and 10 years, respectively, are exactly twice as large as in the HJM base case parameters set and, hence, twice as large as in the Vasicek example. The speculative demands for stocks and bonds are similar to those in the base case, i.e. a logarithmic utility investor invests an 80% fraction of wealth in the stock, 0% in bonds, and 20% in the bank account.

The parameter set in (39) differs from the base set of parameters in (37) alone by having zero prices on risk so that the speculative demands for stocks and bonds are zero, i.e. a logarithmic utility investor in this case invests a 0% fraction of wealth in the stock, 0% in bonds, and 100% in the bank account.

The optimal forward-expected consumption patterns in the HJM three-factor example with the above parameter choices are tabulated in Table 3 under the labels "HJM-2" and "HJM-3", respectively. The optimal forward-expected consumption patterns for the benchmark parameter set in (37) are identical to those in Table 2 and tabulated under the label "HJM-1" in Table 3. For the polar cases of log-utility investors and infinitely risk averse investors the optimal consumption patterns are unaltered across the different parameter sets since they only depend on the initial form of the term structure; these cases are, therefore, not tabulated in Table 3.

For investors with preferences in between the polar cases of logarithmic utility and infinite risk aversion, the forward-expected consumption patterns depend on the specific set of parameters applied, as can be seen from Table 3. Nevertheless, it seems that the optimal consumption patterns do not change dramatically across the different parameter sets. In particular, the consumption patterns in the case of higher forward rate volatilities are basically similar to those in the benchmark parameter case (37) and in the Vasicek example.

In order to have an objective measure of the distance between the different consumption plans in Table 3 and, hence, of the relevant bonds to hedge against changes in the opportunity set, we have also tabulated Fisher–Weil durations in Table 3. The Fisher–Weil duration measure is in this context defined by

$$\frac{\int_t^T (s-t)k(s)P_t(s)\,\mathrm{d}s + (T-t)k(T)P_t(T)}{\int_t^T k(s)P_t(s)\,\mathrm{d}s + k(T)P_t(T)}$$

and is a measure of the average time to the payments of any particular bond. Even for the case of zero risk premia, the durations of the relevant coupon bonds for

hedging against changes in the opportunity set seem close to the relevant durations implied by the other parameter sets considered in Table 3.

5. Conclusion

This paper has provided a characterization of the solution to a general intertemporal consumption and investment problem in a dynamically complete market. We have provided explicit results showing how to hedge against changes in the investment opportunity set in the case of multi-factor Gaussian HJM interest rates and deterministic market prices of risk. In particular, it was demonstrated that changes in the investment opportunity set can be hedged by a single bond: a zero-coupon bond for the case of utility from terminal wealth only and a continuous-coupon bond that equals the forward-expected consumption pattern in the case of utility from intermediate consumption. Explicit numerical examples featuring non-Markovian term structure dynamics suggested that the current form of the term structure of interest rates is important for the optimal consumption pattern and, hence, has important consequences for the appropriate hedge bond, while the specific dynamics of the term structure are of minor importance.

Acknowledgements

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